



**IN THE NAME OF ALLAH,  
MOST GRACIOUS, MOST MERCIFUL**



# COMPLEX ANALYSIS

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## *Dedication*

*In the memory of my parents, to my wife Radhia,  
my son Eymen, my daughters Sondes, Feinel, Asma and  
Tessim and their admirable families. With my love for all.*

*Mathematics is the alphabet with which  
God has written the universe*

*Galileo Galilei*

# Preface

This book is based on courses of lectures that I gave at Monastir University and King Saud University. It covers standard complex analysis material while addressing other related topics. The reader is assumed to have knowledge of the elements of the theory of functions of a real variable. A valuable aspect of this book also is the incorporation of a large number of exercises and problems with detailed solutions.

Master's students will be able to start from Chapter 3.

I would like to thank a number of people who volunteered their time and energy in reviewing one or more chapters: Hassine Elmir, Jamel Benameur, Akhlaq Siddiqui and the unknown reviewer. I express my warmest thanks to the Mathematics Department at College of Sciences, King Saud University.

I conclude this preface with a personal tribute to all my close colleagues at the universities of Monastir and King Saud University; including Professor Khalifa Harzallah, where the lesson plan is mostly rooted in his course.

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*October /1/ 2019*

MONGI BLEL





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